4.6 – Dimension

Theorem 4.6.2 Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for V.
a) If a set in V has more than n vectors, then it is linearly dependent.
b) If a set in V has fewer than n vectors, then it does not span V.

Theorem 4.6.1 All bases for a finite-dimensional vector space have the same number of vectors.

Definition: The **dimension** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V (in some physical contexts, "dimension" is referred to as **degrees of freedom**). In addition, the zero vector space is defined to have dimension zero.

Find a basis for the solution space of each homogeneous linear system, and find the dimension of that space.

#1

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

#5 $\begin{aligned} 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$

#8 In each part, find a basis for the given subspace of R^4, and state its dimension.

a. All vectors of the form $(a,\ b,\ c,\ 0)$.

b. All vectors of the form $(a,\ b,\ c,\ d)$, where $d = a + b$ and $c = a - b$.

c. All vectors of the form $(a,\ b,\ c,\ d)$, where $a = b = c = d$.

Theorem 4.6.4 Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

Theorem 4.6.5 Let S be a finite set of vectors in a finite-dimensional vector space V.
a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

#12 Find standard basis vectors for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.
a. $\mathbf{v}_1 = (-1, 2, 3)$, $\mathbf{v}_2 = (1, -2, -2)$
b. $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (3, 1, -2)$

#18 Find a basis for the subspace of R^4 that is spanned by the vectors $\mathbf{v}_1 = (1, 1, 1, 1)$, $\mathbf{v}_2 = (2, 2, 2, 0)$, $\mathbf{v}_3 = (0, 0, 0, 3)$, $\mathbf{v}_4 = (3, 3, 3, 4)$.

#20 In each part, let T_A be multiplication by A and find the dimension of the subspace of R^4 consisting of all vectors x for which $T_A(\mathbf{x}) = \mathbf{0}$.

a. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$